A NEW ANALYTICAL MODEL FOR STRESS CONCENTRATION AROUND HARD SPHERICAL PARTICLES IN METAL MATRIX COMPOSITES

A Senior Scholars Thesis

by

MATTHEW WADE HARRIS

Submitted to the Office of Undergraduate Research
Texas A&M University
in partial fulfillment of the requirements for the designation as

UNDERGRADUATE RESEARCH SCHOLAR

April 2007

Major: Mechanical Engineering

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Associate Dean for Undergraduate Research: Robert C. Webb

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ABSTRACT

A New Analytical Model for Stress Concentration around Hard Spherical Particles in Metal Matrix Composites (April 2007)

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This analytical model predicts the stress concentration around an elastic, spherical particle in an elastic-plastic metal matrix using strain gradient plasticity theory and a finite unit cell. The model reduces to the special case with a spherical particle in an infinite matrix. It simplifies to models based on classical elasticity and plasticity, also. The solution explains the particle size effect and accounts for composites with dilute and non-dilute particle distributions. Numerical results show that the stress concentration factor is small when the particle size is tens of microns. The stress concentration factor approaches a constant when the particle size is greater than 200 microns.

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CHAPTER I

INTRODUCTION: RESEARCH IMPORTANCE

Ceramic particle reinforced aluminum metal matrix composites (MMCs) are lightweight, strong, thermally stable, and cost-effective (e.g., Lloyd, 1994; Chawla et al., 2001; Miracle, 2005). However, hard, brittle ceramic particles in a ductile matrix induce stress concentrations at the particle-matrix interface leading to particle breaking and interface debonding. These are two leading void/crack nucleation mechanisms associated with MMC fracture. Hence, understanding stress concentrations around brittle, elastic particles in a ductile, elastic-plastic metal matrix is important.

Past studies show that the stress concentration factor at the particle-matrix interface decreases as remote stress triaxiality increases and the strain hardening level decreases (e.g., Wilner, 1988). Existing stress concentration models (e.g., Thomson, 1984; Wilner, 1988) cannot capture the experimentally observed particle size effect. These models are numerical and use an infinitely large matrix, which is only accurate for composites with a small particle volume fraction, i.e., a dilute particle distribution.

This analytical model explains the particle size effect and accounts for dilute and non-dilute particle distributions using a strain gradient plasticity theory and a finite unit cell. The model yields a closed-form solution containing an internal material length scale.

This thesis follows the format of the International Journal of Solids and Structures.

The solution simplifies to the special case with an infinitely large matrix and gives the stress concentration analytically. Numerical results illustrate the derived formulas' application and compare with existing models.

CHAPTER II

BOUNDARY VALUE PROBLEM AND SOLUTION

Classical plasticity theories lack a material length scale and cannot interpret size effect (e.g., Hutchinson, 2000). The strain gradient plasticity theory elaborated by Mühlhaus and Aifantis (1991) introduces higher-order strain gradients into the yield condition. This theory's simplest version uses

$$\sigma_e = \sigma_e^H - c\nabla^2 \varepsilon_e \tag{1}$$

in the yield criterion, where σ_e and σ_e^H are the total and the homogeneous part of the effective stress, ε_e is the effective plastic strain, ∇^2 is the Laplacian operator, and c is the gradient coefficient. This coefficient is a force-like constant measuring the strain gradient effect, which can be positive or negative depending on the material's microstructure.

The extra boundary conditions from the strain gradient term in Eq. (1) are

$$\frac{\partial \varepsilon_e}{\partial m} = 0 \qquad \text{and} \qquad \varepsilon_e = \overline{\varepsilon_e} \qquad \text{on} \;\; \partial^P B. \tag{2}$$

$\partial^P B$ is the plastic boundary, m is the unit outward normal to $\partial^P B$, and the over-bar stands for a prescribed value. The formulation below uses Eq. (1) and Eq. (2) and Hencky's deformation theory of plasticity.

Formulation

The finite unit cell is a spherical matrix region with outer radius b and inner radius a. A spherical particle with radius a is concentric with the matrix region. Fig. 1 shows the hydrostatic tension, σ_0, applied to the matrix outer surface where $r = b$. The matrix and the particle materials are homogeneous and isotropic.

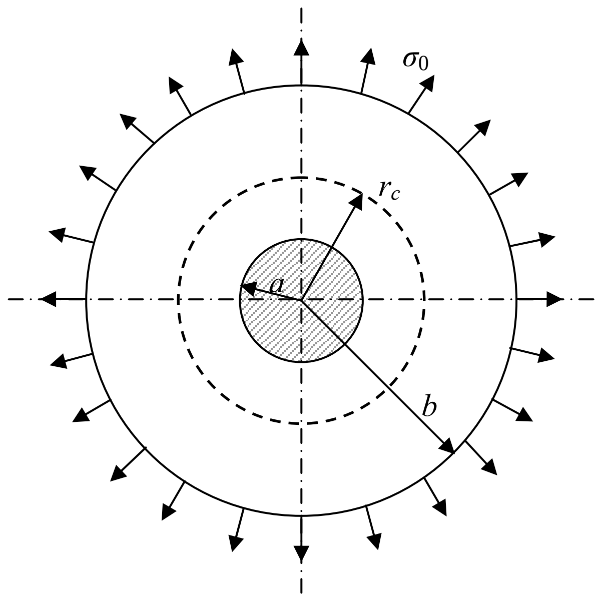

Fig. 1. Problem configuration.

The particle bonds perfectly to the elastic-plastic matrix with interface tension, p_i, and behaves elastically under $\sigma_{0.}$ A classical plasticity model (e.g., Wilner, 1988) uses the same hydrostatic loading and a similar unit cell (with $b \to \infty$).

The entire matrix remains elastic when σ_0 is sufficiently small. When σ_0 becomes large enough the matrix yields from its inner surface because the hard particle induces a stress concentration. The yielded region expands as σ_0 continues to increase. From symmetry, the elastic-plastic interface in the matrix is a spherical surface for any σ_0 that produces a plastic region.

The elasto-plastic radius is r_c and the associated interface tension is p_c. Thus, the matrix material within $a \le r \le r_c$ is plastic and the material within $r_c \le r \le b$ remains elastic under σ_0.

Eqs. (3a,b) show the elastic power-law hardening material in a complex stress state (e.g., Gao, 1992, 2003).

$$\sigma_e^H = \begin{cases} E\varepsilon_e & (\sigma_e \le \sigma_y) \\ \kappa\varepsilon_e^n & (\sigma_e > \sigma_y) \end{cases} \tag{3a,b}$$

E is Young’s modulus, n ($0 \le n \le 1$) is the strain-hardening exponent, σ_y is the yield stress, κ is a material constant satisfying $\kappa = \sigma_y^{1-n}E^n$. Eqs. (3a,b) recover the stress-strain relation for elastic-perfectly plastic materials when $n = 0$. They reduce to Hooke’s law for linearly elastic materials when $n = 1$.

This constitutive model describes the matrix material. Moreover, Eq. (3b) is the homogeneous part of the effective stress, σ_e^H, in the strain gradient plasticity theory in Eq. (1) and Eq. (2). The solution in the plastic region uses this relationship. The material response in the elastic region obeys Hooke's law, Eq. (3a). This enables the direct application of Lamé's classical elasticity solution in the elastic region.

For *infinitesimal* deformations considered in the current formulation, the boundary conditions at the *perfectly bonded* particle-matrix interface are

$$\sigma_{rr}^M\Big|_{r=a} = \sigma_{rr}^I\Big|_{r=a} \quad \text{and} \quad u^M\Big|_{r=a} = u^I\Big|_{r=a}. \tag{4a,b}$$

The superscripts M and I denote the matrix and inclusion, respectively. σ_{rr} is the radial stress component and u is the only non-vanishing radial displacement component. Eqs. (4a,b) ensure the traction and displacement continuities at the interface where $r = a$.

The elastic-plastic problem is now a boundary-value problem with an analytical solution.

Solution for the elastic inclusion $(0 \le r \le a)$

The inclusion is an elastic, solid sphere with radius a subjected to the uniform tension, p_i, normal to its surface. Lamé's solution for a pressurized spherical shell (e.g., Timoshenko, 1970) gives the stress components as

$$\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{\phi\phi} = p_i\,, \tag{5}$$

and the displacement component as

$$u = \frac{1-2\nu^I}{E^I}\,p_i r. \tag{6}$$

E^I and ν^I denote the inclusion's elastic modulus and Poisson's ratio, respectively. p_i is a constant parameter, i.e., it depends on σ_0 and material properties. Eq. (5) shows that the inclusion is in a constant stress state.

Solution for the matrix in the elastic region $(r_c \le r \le b)$

This region is a thick-walled spherical shell with inner radius, r_c, and outer radius, b. The internal tension, p_c, and external tension, σ_0, act on the region. Lamé's solution for a pressurized spherical shell (e.g., Timoshenko, 1970) yields the stress components as

$$\begin{aligned}\sigma_{rr} &= \frac{\sigma_0 b^3}{b^3 - r_c{}^3}\left(1 - \frac{r_c^3}{r^3}\right) + \frac{p_c r_c{}^3}{b^3 - r_c{}^3}\left(\frac{b^3}{r^3} - 1\right),\\ \sigma_{\theta\theta} = \sigma_{\phi\phi} &= \frac{\sigma_0 b^3}{b^3 - r_c{}^3}\left(1 + \frac{r_c^3}{2r^3}\right) - \frac{p_c r_c{}^3}{b^3 - r_c{}^3}\left(1 + \frac{b^3}{2r^3}\right),\end{aligned} \tag{7}$$

and the displacement component as

$$u = \frac{r}{E}\left\{(1-\nu)\left[\frac{\sigma_0 b^3}{b^3 - r_c{}^3}\left(1 + \frac{r_c^3}{2r^3}\right) - \frac{p_c r_c{}^3}{b^3 - r_c^3}\left(1 + \frac{b^3}{2r^3}\right)\right] - \nu\left[\frac{\sigma_0 b^3}{b^3 - r_c{}^3}\left(1 - \frac{r_c^3}{r^3}\right) + \frac{p_c r_c{}^3}{b^3 - r_c{}^3}\left(\frac{b^3}{r^3} - 1\right)\right]\right\}. \tag{8}$$

The solution in Eq. (7) and Eq. (8) contains two unknown parameters, p_c and r_c.

On the elastic-plastic interface where $r = r_c$, the stress components in Eq. (7) must satisfy the yield condition

$$\sigma_e \,|_{r=r_c} = \sigma_y \,. \tag{9}$$

This provides the first relation for determining p_c and r_c.

Solution for the matrix in the plastic region $(a \le r \le r_c)$

The governing equations below assume infinitesimal deformations, isotropic hardening, incompressibility, and monotonic loading. These equations embody Hencky's deformation theory, strain gradient plasticity theory, and the elastic power-law hardening model. The governing equations include the equilibrium equation,

$$\sigma_{\theta\theta} - \sigma_{rr} = \frac{1}{2} r \frac{d\sigma_{rr}}{dr}; \tag{10}$$

the compatibility equation,

$$r \frac{d\varepsilon_{\theta\theta}}{dr} = \varepsilon_{rr} - \varepsilon_{\theta\theta}; \tag{11}$$

and the constitutive equations,

$$\varepsilon_{rr} = -\frac{\varepsilon_e}{\sigma_e}(\sigma_{\theta\theta} - \sigma_{rr}), \quad \varepsilon_{\theta\theta} = \frac{1}{2}\frac{\varepsilon_e}{\sigma_e}(\sigma_{\theta\theta} - \sigma_{rr}) = \varepsilon_{\phi\phi}, \tag{12}$$

$$\sigma_e = \kappa \varepsilon_e{}^n - c\nabla^2 \varepsilon_e, \tag{13}$$

$$\sigma_e = \sigma_{\theta\theta} - \sigma_{rr}. \tag{14}$$

The boundary conditions are

$$\sigma_{rr}\,|_{r=a} = p_i, \quad \sigma_{rr}\,|_{r=r_c} = p_c, \tag{15a,b}$$

$$\varepsilon_e\,|_{r=a} = D, \quad \varepsilon_e\,|_{r=r_c} = \frac{\sigma_y}{E}, \tag{16a,b}$$

where D is a constant. Eqs. (15a,b) are two standard boundary conditions in classical plasticity. Eqs. (16a,b) are two extra boundary conditions arising from strain gradient plasticity theory.

Eq. (10) to Eq. (16a,b) defines the boundary-value problem (BVP) determining the stress and displacement components in the plastic region. The solution gives the stress components as

$$\sigma_{rr} = \sigma_0 - \frac{2\sigma_y}{3}\left(1-\frac{r_c^{\,3}}{b^3}\right) + 2\left[\frac{\sigma_y}{3n}\left(1-\frac{r_c^{\,3n}}{r^{3n}}\right) - \frac{6}{5}\frac{c}{a^2}\left(\frac{\sigma_y}{E}\right)\left(\frac{r_c}{a}\right)^3\left(\frac{a^5}{r_c^{\,5}}-\frac{a^5}{r^5}\right)\right],$$
$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = \sigma_0 - \frac{2\sigma_y}{3}\left(1-\frac{r_c^{\,3}}{b^3}\right) + 2\left\{\frac{\sigma_y}{3n}\left[1+\left(\frac{3n}{2}-1\right)\left(\frac{r_c}{r}\right)^{3n}\right] - \frac{6}{5}\frac{\sigma_y}{E}\frac{c}{a^2}\left(\frac{r_c}{a}\right)^3\left(\frac{a^5}{r_c^{\,5}}+\frac{3}{2}\frac{a^5}{r^5}\right)\right\},$$

(17)

and the displacement component as

$$u = \frac{1}{2}\frac{\sigma_y}{E}\frac{r_c^{\,3}}{r^2}. \qquad (18)$$

Eq. (19) defines r_c as

$$\frac{1}{2}\frac{E^I}{1-2v^I}\frac{\sigma_y}{E}\left(\frac{r_c}{a}\right)^3 = \sigma_0 - \frac{2\sigma_y}{3}\left(1-\frac{r_c^{\,3}}{b^3}\right) - 2\left[\frac{\sigma_y}{3n}\left(\frac{r_c}{a}\right)^{3n}\left(1-\frac{a^{3n}}{r_c^{\,3n}}\right) - \frac{6}{5}\frac{c}{a^2}\left(\frac{\sigma_y}{E}\right)\left(\frac{r_c}{a}\right)^3\left(1-\frac{a^5}{r_c^{\,5}}\right)\right] \qquad (19)$$

for given values σ_0, E, σ_y, n, c, E^I, v^I, a and b. The remaining three parameters are

$$p_i = \frac{1}{2}\frac{E^I}{1-2v^I}\frac{\sigma_y}{E}\left(\frac{r_c}{a}\right)^3, \quad p_c = \sigma_o - \frac{2\sigma_y}{3}\left(1-\frac{r_c^{\,3}}{b^3}\right), \quad D = \frac{\sigma_y}{E}\frac{r_c^{\,3}}{a^3}. \qquad \text{(20a–c)}$$

The stress and displacement components for the inclusion now come from Eq. (5) and Eq. (6). Eq. (7) and Eq. (8) give the components in the elastic region.

CHAPTER 3

SPECIFIC SOLUTIONS

Classical plasticity solution

Eq. (10) to Eq. (16a,b) defines the BVP in the plastic region. These equations reduce to formulas from Hencky deformation theory and the von Mises yield criterion when c = 0. Hence, letting c = 0 in Eq. (17) gives the stress components.

$$\begin{aligned}\sigma_{rr} &= \sigma_o - \frac{2\sigma_y}{3}\left(1-\frac{{r_c}^3}{b^3}\right)+\frac{2\sigma_y}{3n}\left(1-\frac{{r_c}^{3n}}{r^{3n}}\right)\\ \sigma_{\theta\theta} = \sigma_{\phi\phi} &= \sigma_o - \frac{2\sigma_y}{3}\left(1-\frac{{r_c}^3}{b^3}\right)+\frac{2\sigma_y}{3n}\left[1+\left(\frac{3n}{2}-1\right)\left(\frac{r_c}{r}\right)^{3n}\right]\end{aligned} \tag{21}$$

Eq. (19) reduces to Eq. (22) and gives r_c.

$$\frac{1}{2}\frac{E^I}{1-2\nu^I}\frac{\sigma_y}{E}\left(\frac{r_c}{a}\right)^3 = \sigma_o - \frac{2\sigma_y}{3}\left(1-\frac{{r_c}^3}{b^3}\right)-\frac{2\sigma_y}{3n}\left(\frac{r_c}{a}\right)^{3n}\left(1-\frac{a^{3n}}{{r_c}^{3n}}\right) \tag{22}$$

Inclusion in an infinitely large elastic-plastic matrix

The elastic-plastic matrix becomes infinitely large as b approaches infinity. Letting $b \to \infty$ in Eq. (17) gives the stress components.

$$\begin{aligned}\sigma_{rr} &= \sigma_0 - \frac{2\sigma_y}{3} + 2\left[\frac{\sigma_y}{3n}\left(1-\frac{{r_c}^{3n}}{r^{3n}}\right)-\frac{6}{5}\frac{c}{a^2}\left(\frac{\sigma_y}{E}\right)\left(\frac{r_c}{a}\right)^3\left(\frac{a^5}{{r_c}^5}-\frac{a^5}{r^5}\right)\right]\\ \sigma_{\theta\theta} = \sigma_{\phi\phi} &= \sigma_0 - \frac{2\sigma_y}{3} + 2\left\{\frac{\sigma_y}{3n}\left[1+\left(\frac{3n}{2}-1\right)\left(\frac{r_c}{r}\right)^{3n}\right]-\frac{6}{5}\frac{\sigma_y}{E}\frac{c}{a^2}\left(\frac{r_c}{a}\right)^3\left(\frac{a^5}{{r_c}^5}+\frac{3}{2}\frac{a^5}{r^5}\right)\right\}\end{aligned} \tag{23}$$

Solving Eq. (24) gives r_c.

$$\frac{1}{2}\frac{E^I}{1-2v^I}\frac{\sigma_y}{E}\left(\frac{r_c}{a}\right)^3=\sigma_0-\frac{2\sigma_y}{3}-2\left[\frac{\sigma_y}{3n}\left(\frac{r_c}{a}\right)^{3n}\left(1-\frac{a^{3n}}{{r_c}^{3n}}\right)-\frac{6}{5}\frac{c}{a^2}\left(\frac{\sigma_y}{E}\right)\left(\frac{r_c}{a}\right)^3\left(1-\frac{a^5}{{r_c}^5}\right)\right] \tag{24}$$

CHAPTER 4

CONCLUSIONS: STRESS CONCENTRATION FACTOR

The stress concentration factor, K_t, on the inclusion/matrix interface is the interfacial normal stress to the applied (hydrostatic) tension ratio (e.g., Wilner, 1988).

$$K_t \equiv \frac{\sigma_{rr}^{M}\big|_{r=a}}{\sigma_0} \tag{25}$$

Substituting Eq. (15a) and Eq. (20a) into Eq. (25) gives

$$K_t = \frac{\sigma_y}{\sigma_0}\frac{E^I}{2\left(1-v^I\right)E}\left(\frac{r_c}{a}\right)^3, \tag{26}$$

where Eq. (19) provides r_c. Eq. (26) is valid for the general case involving an elastic-plastic matrix and an elastic inclusion.

Eq. (27) gives the stress concentration factor when the matrix is entirely elastic.

$$K_t = \frac{\dfrac{3}{2}\dfrac{b^3}{b^3-a^3}}{\dfrac{E}{E^I}\dfrac{1-2v^I}{1-v}+\dfrac{v}{1-v}+\dfrac{2a^3+b^3}{2(b^3-a^3)}} \tag{27}$$

This closed-form expression shows that K_t varies with the elastic matrix properties, E and v; the elastic inclusion properties, E^I and v^I; and the unit cell geometry, a and b.

Eq. (27) reduces to Eq. (28) when the matrix is infinitely large.

$$K_t = \frac{3(1-v)}{1+v+2(1-2v^I)\frac{E}{E^I}} \tag{28}$$

This is identical to the solution provided by Wilner (1988).

Fig. 2. presents numerical results to illustrate the solution. The material properties are E = 68 GPa, n = 0.25, and c = − 2.5 N for an aluminum matrix and E^I = 401 GPa and v^I = 0.22 for a SiC particle. The particle volume fraction defined by

$$\phi = \frac{\frac{4}{3}\pi a^3}{\frac{4}{3}\pi b^3} = \frac{a^3}{b^3} \tag{29}$$

is 5%. Eq. (26) and Eq. (19) yield the numerical values appearing in Fig. 2 for the material and geometrical properties above.

Fig. 2 shows the stress concentration factor depends on particle size. The stress concentration factor is small when the reinforcing particle is very small (tens of microns). This explains the size, or strengthening, effect at the micron scale. The stress concentration factor approaches a constant when the particle size is large (greater than 200 microns). Hence, the stress concentration factor is particle size dependent.

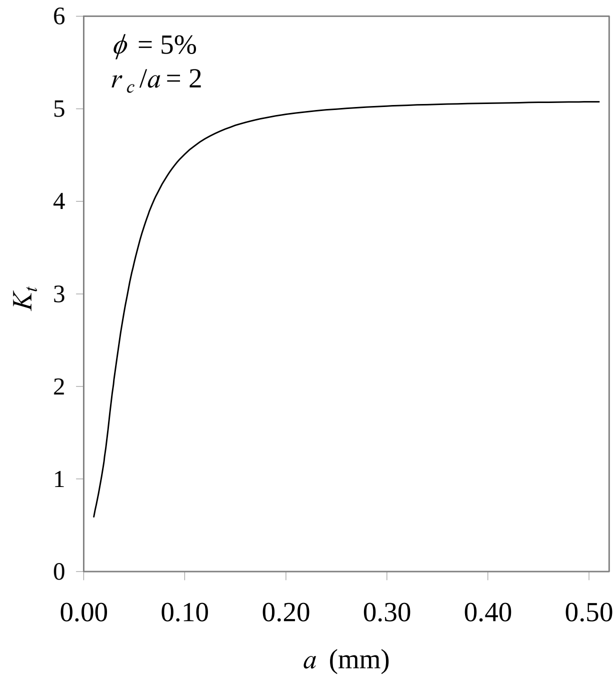

Fig. 2. Stress concentration factor as a function of the inclusion size.

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